Regularization for 3D DC resistivity inversion

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SUMMARY

We present current research associated with a common project of the TU Bergakademie Freiberg and the Deutsches Geoforschungszentrum Potsdam (GFZ). The aim of this project is to exploit the distinct sensitivity patterns of different EM methods to enhance the resolution.

Here, we focus on the DC resistivity method. We have reimplemented our finite element secondary field DC forward modeling approach on unstructured grids and proved its theoretical convergence rate of $\mathcal{O}(h^2)$.
To stabilise the inversion procedure and provide additional information to avoid ambiguities, a suitable regularization strategy is necessary. We have implemented a smoothness regularization in which the penalty function measures the norm of a weak gradient of the conductivity field. It is implemented using a mixed finite element method with Raviart-Thomas elements on unstructured grids.

Preliminary tests using a synthetic conductivity model and a small number of electric sources yield promising results in view of the inversion of real DC resistivity data sets.

Keywords: 3D DC resistivity, mixed finite element methods, regularization

INTRODUCTION

The Geotechnologien Project *Three-dimensional Multi-Scale and Multi-Method Inversion to Determine the Electrical Conductivity Distribution of the Subsurface Using Parallel Computing Architectures* (Multi-EM) addresses the combination of different electromagnetic methods in a joint inversion approach to exploit their individual advantages. In Freiberg, we focus on the combination of the transient electromagnetic (TEM) and the DC resistivity method.

We have implemented a new DC resistivity forward operator in MATLAB using finite elements on unstructured tetrahedral grids that can easily be combined with our already existing TEM software (Afanasjew et al., 2010). This code enables us to deal with even complex topography and to extract the derivatives, which are crucial for the inversion while retaining full control over the assembly process of the system matrix (Weißflog et al., 2012).

For simplicity, we apply a regularized Gauss-Newton method in view of a combination of different electromagnetic methods in one inversion algorithm. We focus on an appropriate regularization technique that has been outlined by Schwarzbach and Haber (2012).

3D DC RESISTIVITY MODELING

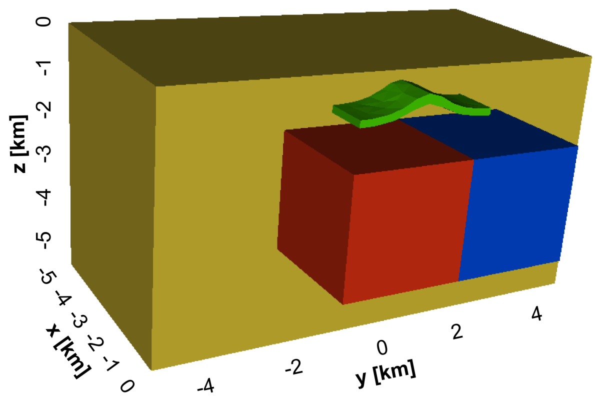

Figure 1: Synthetic model for the Multi-EM project, currently used in Freiberg for DC inversion with $\sigma_1 = 0.1\,\frac{\mathrm{S}}{\mathrm{m}}$ (yellow), $\sigma_2 = 0.01\,\frac{\mathrm{S}}{\mathrm{m}}$ (green), $\sigma_3 = 1\,\frac{\mathrm{S}}{\mathrm{m}}$ (red) and $\sigma_4 = 0.002\,\frac{\mathrm{S}}{\mathrm{m}}$ (blue).

DC resistivity modeling requires the discretisation of the equation of continuity:

$$-\nabla \cdot (\sigma \nabla \phi) = I\delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{1}$$

with ϕ as the electric potential, a given distribution of conductivity $\sigma(\boldsymbol{x})$ (cf. Figure 1) and a point source (electrode) A of strength I located at $\boldsymbol{x}_0 \in \mathbb{R}^3$.

Finite elements enable us to deal with complex structures and permit the extraction of the derivatives with respect to the model parameters from the system matrix $A(\boldsymbol{\sigma})$ in a straightforward manner.

SMOOTHNESS REGULARIZATION

The inversion process requires us to solve a minimization problem which combines the residual norm and a regularization operator $R(\boldsymbol{m})$:

$$\Phi(\boldsymbol{m}) = \|Q\boldsymbol{u} - \boldsymbol{b}\|_2^2 + \alpha R(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}}) \to \min_{\boldsymbol{m}} \quad (2)$$

subject to $A(\boldsymbol{m})\boldsymbol{u} = \boldsymbol{f}$. Here, $\boldsymbol{b}$ is the measured data, $\boldsymbol{u} = A(\boldsymbol{m})^{-1}\boldsymbol{f}$ the modeled data, $\boldsymbol{m}$ are the model parameters, Q is some measurement operator ($Q\boldsymbol{u} \approx \boldsymbol{b}$), $\boldsymbol{m}_{\text{ref}}$ is the reference model and α represents the regularization parameter.

To stabilise the inversion procedure and provide additional information to avoid ambiguities, a suitable regularization strategy is necessary. As our inversion approach is based on a finite element discretisation of the potential equation (1) using a piecewise constant representation of the conductivity model, this requires a regularization operator $R(\boldsymbol{m})$ applicable to piecewise constant model parameters on unstructured grids and weak formulations. We have implemented a smoothness regularization in which the penalty function measures the norm of a weak gradient of the conductivity field:

$$R(\boldsymbol{m}) = \int_\Omega \nabla\boldsymbol{m} \cdot \nabla\boldsymbol{m}\, dV. \quad (3)$$

As laid out in Brezzi and Fortin (1991) we derive the mixed formulation of (2) incorporating (3):

$$\Phi(\boldsymbol{m},\boldsymbol{u}) = F(\boldsymbol{m}) - \alpha \int_\Omega |\boldsymbol{u}|^2\, dV + 2\alpha \int_\Omega (\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})\nabla\cdot\boldsymbol{u}\, dV, \quad (4)$$

with $F(\boldsymbol{m}) = \|Q\boldsymbol{u} - \boldsymbol{b}\|_2^2$ being the residual norm, $\boldsymbol{m} \in L^2(\Omega)$ and $\boldsymbol{u} \in H_0(\text{div};\Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3; \nabla\cdot\boldsymbol{u} \in L^2(\Omega); \boldsymbol{n}\cdot\boldsymbol{u}|_{\partial\Omega}\}$. To achieve a conformal discretization we use Raviart-Thomas elements of lowest order (RT_0). This ensures continuity of normal components between elements. A discrete representation of (4) then reads:

$$\Phi(\boldsymbol{m},\boldsymbol{u}) = F(\boldsymbol{m}) - \alpha\boldsymbol{u}^T M_{RT_0}\boldsymbol{u} + 2\alpha(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})^T D\boldsymbol{u}. \quad (5)$$

M_{RT_0} is the RT_0 mass matrix and D is the discrete divergence operator. In order to minimize (5) the gradient of $\Phi(\boldsymbol{m},\boldsymbol{u})$ with respect to $\boldsymbol{u}$ must vanish:

$$\begin{aligned} -2\alpha M_{RT_0}\boldsymbol{u} + 2\alpha D^T(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}}) &= \boldsymbol{0} \\ &\Downarrow \\ \boldsymbol{u} = M_{RT_0}^{-1} D^T(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}}). & \quad (6) \end{aligned}$$

Using condition (6) we eliminate the dual variable and arrive at the final objective function:

$$\Phi(\boldsymbol{m}) = F(\boldsymbol{m}) - \alpha(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})^T \cdot D M_{RT_0}^{-1} D^T(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}}) \to \min_{\boldsymbol{m}}. \quad (7)$$

Applying a Gauss-Newton scheme to solve this non-linear least squares problem enables us to linearize $\Phi(\boldsymbol{m})$. The resulting linear problem is solved by an iterative Krylov subspace method to find a better approximation to the model parameters $\boldsymbol{m}$ in each Gauss-Newton step.

NUMERICAL EXPERIMENTS

To judge the effectiveness of our regularization approach, we tried to reconstruct the model shown in figure 1 from synthetic data.

As reference model $\boldsymbol{m}_{\text{ref}}$ a homogeneous conductivity of $\sigma = 0.1\,\frac{\text{S}}{\text{m}}$, which is equal to the conductivity at the source, was used. This is also what we used as our starting model $\boldsymbol{m}_0$. The model was coarsly discretized into 12903 elements and allowed the parameter $\boldsymbol{m}$ to vary on each tetrahedron. The following figure shows a frontal slice of the model at $x = 0\,\text{m}$.

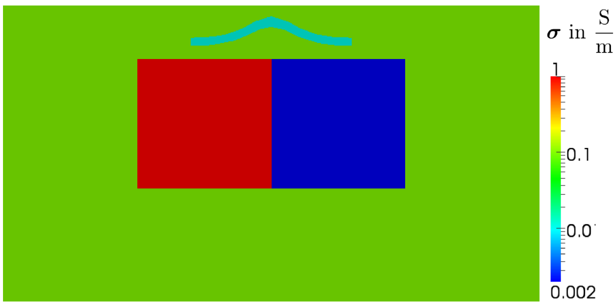

Figure 2: Synthetic model

We present inversion results for two different numbers of sources located on the earth's surface. Each source generates 1953 data points.

The regularization parameter α was chosen to be $\alpha = \max\{10^{-4-k}, 10^{-14}\}$ in Gauss-Newton step k as a constant value of α resulted in less accurate reconstruction of our model. The iteration was terminated once we reached a residual norm of 10^{-7}.

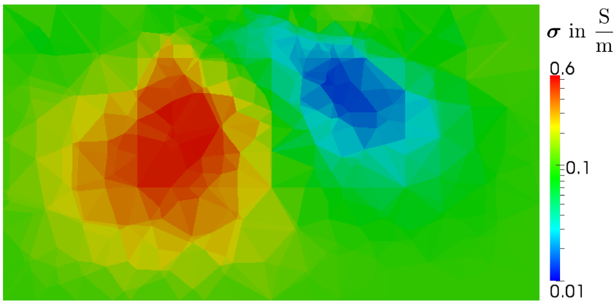

Figure 3: Inversion result for two sources

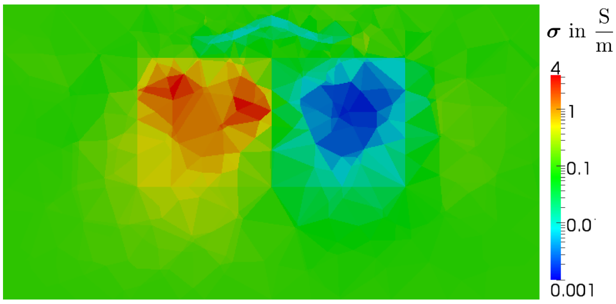

Figure 4: Inversion result for nine sources

Figures 3 and 4 show the inversion results for two and nine sources. The regularization operator stabilized the iteration and guaranteed the convergence of the Gauss-Newton scheme.

As the number of sources is increased, the reconstruction of embedded conductive or resistive bodies becomes progressively more accurate.

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