Finite volume modelling of electromagnetic data using unstructured staggered grids

Hormoz Jahandari[1], and Colin G. Farquharson[1]
[1]Memorial University of Newfoundland, St. John's, NL, Canada

SUMMARY

Unstructured schemes possess important advantages in the modelling of total electromagnetic fields where the source location is required to be highly refined, and in the simulation of topographic features of the terrain and geological interfaces. This study investigates the application of the well-known Yee's staggered scheme to unstructured Delaunay-Voronoï grids. The Helmholtz equation for the electric field is discretized using a finite volume approach and the problem is solved for the projection of the total electric field along the Delaunay edges. Edge vector basis functions are used for interpolation inside the tetrahedra and for recovering magnetic fields. As examples for this preliminary study, the responses of a homogeneous half-space and a conductive block in a resistive background to magnetic dipole sources are computed and compared with analytical results.

Keywords: electromagnetics, forward modeling, finite difference, finite volume, unstructured grids

INTRODUCTION

Due to the existence of the singularity and the high gradients of the fields at the source locations, it is common in the modelling of controlled-source electromagnetic fields to decompose the total electromagnetic (EM) fields into primary and secondary parts and solve only for the secondary fields caused by anomalous bodies. The primary fields in this approach are calculated analytically based on the expressions for simple background earths. However, this approach can lose its effectiveness in the presence of complex conductivity structures and highly irregular topographic features. Unstructured total field methods can be effective in these situations by local refinement of the grid at the source locations and simulating the terrain irregularities without an extreme increase in the number of cells in the earth model.

One of the unstructured methods which is the direct generalization of the well-known Yee's scheme (Yee, 1966) is the staggered method proposed by Madsen and Ziolkowski (1990). In this method, the full 3-D vectors of electric and magnetic fields are defined on the edges of the primary and dual cells, respectively. However, when applied to orthogonal staggered grids, this method regains the simplicity of the original Yee's scheme. An important example of such orthogonal grids is the Delaunay-Voronoï configuration, and the finite volume scheme that uses these grids is usually referred to as the "co-volume" method (Hermeline, Layouni, & Omnes, 2008). In this scheme the magnetic field remains divergence free, the method is non-dissipative, and it has been shown to be first-order convergent (Nicolaides & Wang, 1998).

In this preliminary study, the co-volume method is used to solve the Helmholtz equation in order to find the projection of the total electric field along the edges of the tetrahedra present in the grid. Solutions at the observation points are calculated, as a post processing stage, by interpolation using edge vector basis functions of the tetrahedral edges. Two examples are included which present the responses of a homogeneous half-space model and a conductive cube in a resistive half-space due to magnetic dipole sources.

THEORY

In electromagnetic methods, the controlled sources of the EM fields can be either an impressed electric current density, $\mathbf{J}_p$, or magnetic dipole moment, $\mathbf{M}_p$ (Hohmann, 1983). Therefore, at any point the total electric and magnetic fields ($\mathbf{E}$ and $\mathbf{H}$) are related by the following form of Maxwell's equations in the frequency domain:

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H} - i\omega\mu_0\mathbf{M}_p \quad (1)$$

$$\nabla \times \mathbf{H} = \sigma\,\mathbf{E} + \mathbf{J}_p, \quad (2)$$

where σ and μ_0 are conductivity and the permeability of free space, respectively, ω is angular frequency and i is the imaginary unit. Taking the curl of equation 1 and substituting in 2 gives the Helmholtz equation for electric field

$$\nabla\times\nabla\times\mathbf{E}+i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}_p - i\omega\mu_0(\nabla\times\mathbf{M}_p). \quad (3)$$

In this study, this equation is solved for the electric field as a Dirichlet problem which assumes a zero field at infinity.

The co-volume method uses the integral form of Maxwell's equations applied over the Delaunay and Voronoï faces of the grid (an example of the relation between these grids is shown in Figure 1). In order to maintain the continuity of the tangential component of the electric field at the boundary of regions with different conductivities, it is common to define the electric (magnetic) field components along the Delaunay (Voronoï) edges and over Voronoï (Delaunay) faces. Therefore, using Stokes' theorem, Faraday's and Ampère's laws are written as

$$\oint_{\partial A^D} \mathbf{E}\cdot d\mathbf{l}^D = -i\mu_0\omega \iint_{A^D} \mathbf{H}\cdot d\mathbf{A}^D - i\mu_0\omega \iint_{A^D} \mathbf{M}_p\cdot d\mathbf{A}^D \quad (4)$$

$$\oint_{\partial A^V} \mathbf{H}\cdot d\mathbf{l}^V = \sigma \iint_{A^V} \mathbf{E}\cdot d\mathbf{A}^V + \iint_{A^V} \mathbf{J}_p\cdot d\mathbf{A}^V, \quad (5)$$

where the superscripts *D* and *V* signify *Delaunay* and *Voronoï*. A and ∂A are the face and the contour surrounding that face, and **A** and **l** are unit vectors normal to A and along ∂A, respectively. The mutual orthogonality of the Delaunay (Voronoï) edges and Voronoï (Delaunay) faces as well as approximating the fields along the edges and their respective faces by an average constant value allow the following discretizations. By considering the *i*th (*j*th) Delaunay (Voronoï) edge and orthogonal to it the *i*th (*j*th) Voronoï (Delaunay) face, and with *W* showing the number of edges surrounding each face, equations 4 and 5 can be discretized as

$$\sum_{q=1}^{W_j^D} E_{i(j,q)}\, l^D_{i(j,q)} = -i\mu_0\omega H_j\, A_j^D - i\mu_0\omega M_{p_j}\, A_j^D \quad (6)$$

$$\sum_{k=1}^{W_i^V} H_{j(i,k)} l^V_{j(i,k)} = \sigma E_i\, A_i^V + J_{p_i}\, A_i^V, \quad (7)$$

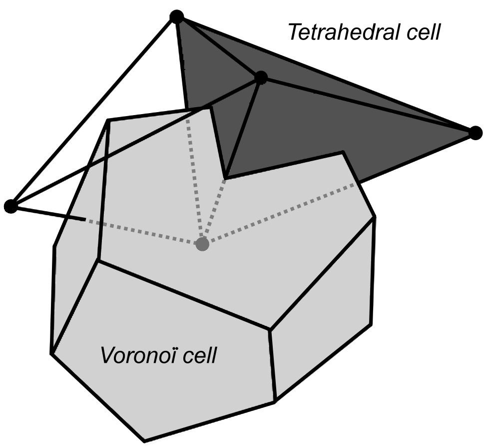

Figure 1. This figure shows the relation between the Delaunay tetrahedral and Voronoï cells. The circumcentres of the tetrahedra form the vertices of the Voronoï cells. The edges of the Delaunay (Voronoï) cells are orthogonal to the Voronoï (Delaunay) faces.

where E_i (H_j) is the projection of the electric (magnetic) field at the intersection of the *i*th (*j*th) Delaunay (Voronoï) edge and its corresponding Voronoï (Delaunay) face along this edge. l is a term with size equal to the length of each edge and a positive or negative sign determined by the mutual orientation of the edges *i* and *j* (see Figure 2 for more details). A global arbitrary orientation is attributed to each edge for this purpose. As the tetrahedral grid is piecewise constant with respect to conductivity, in the relations above σ is the area weighted average of the conductivities of the tetrahedra that share the *i*th Delaunay edge. Moreover, the source terms M_p and J_p are defined at the same locations as the magnetic and electric fields, respectively. The discretized form of the Helmholtz equation can be derived by finding an expression for H from relation 6 and substituting in 7:

$$\sum_{k=1}^{W_i^V} \left(\left(\sum_{q=1}^{W_j^D} E_{i(j,q)}\, l^D_{i(j,q)} \right) \frac{l^V_{j(i,k)}}{A^D_{j(i,k)}} \right) + i\omega\mu_0\sigma E_i\, A_i^V$$

$$= -i\omega\mu_0 \sum_{k=1}^{W_i^V} M_{p_{j(i,k)}} l^V_{j(i,k)} - i\omega\mu_0 J_{p_i}\, A_i^V. \quad (8)$$

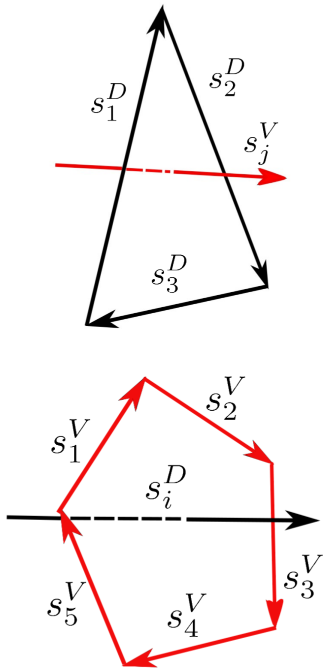

Figure 2. The relation between the *j*th (*i*th) Voronoï (Delaunay) edge and its corresponding Delaunay (Voronoï) face is shown in the top (bottom) figure. The mutual orientation of the Delaunay and Voronoï edges is determined based on their global orientation. In both of the figures this mutual orientation has a negative sign.

This relation can be written for all the N Delaunay edges (or Voronoï faces) of the grid and the system of equations is solved for the unknown electric fields E. In this study the MUMPS sparse direct solver (Amestoy, Guermouche, L'Excellent, & Pralet, 2006) is used to solve this problem. The Dirichlet boundary condition is applied by assigning a zero value to the electric fields on the boundary of the domain.

The solutions at the observation points can be found by interpolation using vector basis functions. At any point inside the grid with coordinates (x, y, z) this interpolated solution is expressed as

$$\mathbf{E}^e(x, y, z) = \sum_{i=1}^{6} \mathbf{N}_i^e(x, y, z) E_i^e, \quad (9)$$

where the $\mathbf{N}_i^e$ terms are the vector basis functions of the edges of the tetrahedron e in which the observation point is located, and E_i^e are the solutions at these edges (Jin, 2002). This relation can also be used to find the magnetic field at an observation point by taking the curl of $\mathbf{E}$ and using Faraday's law.

EXAMPLES

The following examples present the electric and magnetic responses of a homogeneous half-space and a conductive block in a resistive background for magnetic dipole sources. For the both examples the results from the finite volume method are compared with analytic solutions using the expressions of Ward and Hohmann (1988).

The first example is a homogeneous half-space model with the conductivity of 1 *S/m*. An infinitesimal unit magnetic dipole was positioned at $(0, 0, 0)$ (at the earth-air interface) by inserting two regular tetrahedra with the edge size of 2 m. The vertical magnetic dipole was located along the Voronoï edge that passed through the common face of these two tetrahedra and operated at 500 *Hz*. 50 observation points were located along the x axis from -250 to 250 m. For refining the grid at the observation locations, one regular tetrahedron with the edge size of 3 m was inserted for each observation point inside the grid. The entire region with dimensions of $10 \times 10 \times 10$ *km* was tetrahedralized using TetGen (Si, 2004) into 424,697 tetrahedral cells and 493,644 Delaunay edges (which was the number of unknowns). The total computation time and memory usage for solving this problem using MUMPS direct solver were 289 s and 18 *Gbytes*, respectively (on an Apple Mac Pro computer; 2.26 *GHz* Quad-Core Intel Xeon processor). Figure 3 shows the numerical and analytical solutions for the real and imaginary parts of E_y and H_z. Both numerical fields agree well with the analytical data but H_z shows a smoother trend than E_y.

In the second example, a cube with the conductivity of 100 *S/m* was buried in a resistive half-space of 0.0001 *S/m*. The cube had dimensions of 18 m and its top was at $(0, 0, -76)$ m. For this model, the secondary magnetic field was calculated for 15 source-receiver pairs (with 5 m separation) moving along the x axis above the cube and at the interface. The observations were attributed to the middle point of the source-receiver pairs. The sources operat-

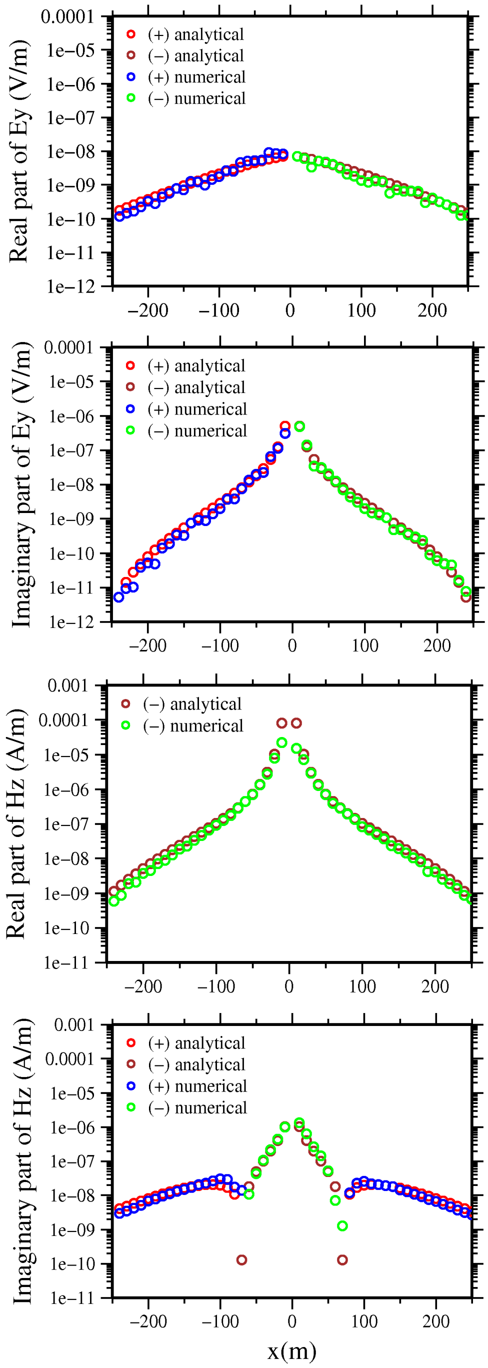

Figure 3. The electric and magnetic responses for the homogeneous half-space model. The source is a vertical magnetic dipole with the frequency of 500 *Hz* which is located at the centre of the profile.

ed at 900 Hz and, like the previous example, regular tetrahedra were inserted inside the grid for refinement at the source and receiver locations (see Figure 4). The region had the same dimensions as the previous example and contained 192, 365 tetrahedral cells. The finite volume results are compared with the analytical solutions due to a sphere with the same conductivity and volume as the cube but in free-space (Figure 5). In spite of the large conductivity contrast, there is a good agreement between the numerical and analytical solutions.

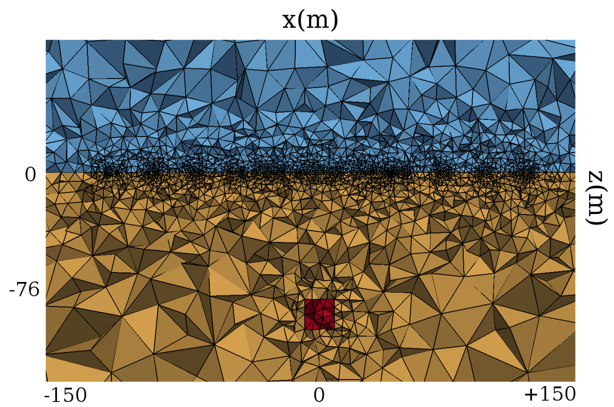

Figure 4. A vertical section of the grid for the second example: a conductive cube (in red color) is buried in a resistive half-space. The grid is refined at the earth-air interface at the location of the source-receiver pairs. The grid is also partially refined at the cube's location.

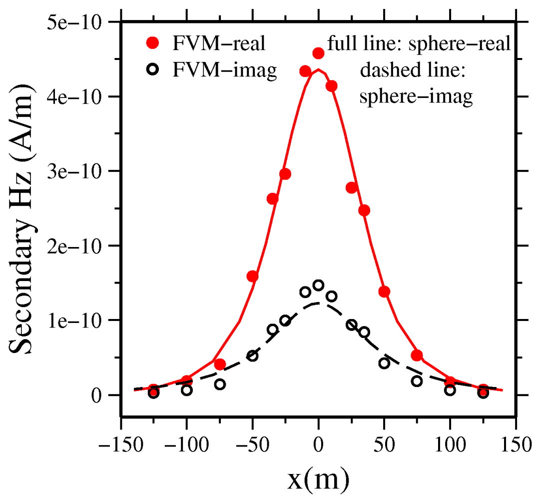

Figure 5. The vertical component of the secondary magnetic field for the source-receiver pairs. Circles show the finite volume results and lines show the analytical solutions. Abscissa gives the middle point of the source-receiver pairs.

CONCLUSIONS

We have developed a staggered finite volume scheme which is a generalization of the Yee algorithm for Delaunay-Voronoï grids. In this method (usually referred to as the "co-volume" method), the electric and magnetic fields are located along the Delaunay and Voronoï edges, respectively, and the Helmholtz equation is solved to find the electric field. In the presented examples, the responses due to a homogeneous half-space and a conductive cube in a resistive background are computed. There is a good agreement between the analytical and numerical results which suggests that the presented method is a candidate for computing EM fields on unstructured grids.

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